DIAGONAL NUCLEAR OPERATORS ON l_p SPACES

BY

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This paper computes the nuclear norm of diagonal nuclear operators from l_q to l_s. If D_α is the diagonal nuclear operator with diagonal entries $\boldsymbol{\alpha} = (a_1, \ldots, a_i, \ldots)$, then the nuclear norm of D_α is $l_h(\alpha)$ where $h = 1$ if $1 \leq s \leq q \leq \infty$, $h = qs/(qs + q - s)$ if $1 \leq q < s < \infty$, $h = q/(q-1)$ if $s = \infty$ and $1 < q < \infty$, $h = \infty$ if $s = \infty$ and $q = 1$. This result depends upon an inequality for bounded operators $A: l_p \to l_r$ which asserts: $\|A\| \geq \|D\|$ where D is the associated diagonal of A.

§§1 and 2 provide the preliminary computations and §3 proves the inequality. The computation of the nuclear norm of diagonal operator is given in §4. The inequality given in §3 has other applications, to be made in a forthcoming paper. Definitions and standard theorems on nuclear operators are from A. Pietsch [2].

§ 1

(1.0) NOTATION. Let p satisfy $1 \leq p < \infty$.

l_p^n will denote the n-dimensional vector space of all sequences $\xi = (\xi_1, \ldots, \xi_n)$ under the norm

$$l_p(\xi) = \left(\sum_{1 \leq i \leq n} |\xi_i|^p \right)^{1/p}.$$

l_p will denote the vector space of all sequences $\xi = (\xi_1, \ldots, \xi_i, \ldots)$ for which the norm

$$l_p(\xi) = \left(\sum_i |\xi_i|^p \right)^{1/p}$$

is finite. Unless otherwise specified, l_p will be taken to be a vector space over the complex numbers.

The dual number of p defined as $(p/(p-1))$ will be denoted by q. When $p = \infty$, we define the dual number q to be 1. The dual number of r is denoted by s.

$\pi_t: l_p \to l_p$ will denote the projection defined by setting

$$\pi_t(\xi_1, \ldots, \xi_i, \ldots) = (\xi_1, \ldots, \xi_t, 0, \ldots).$$

δ_{ij} will denote the Kronecker delta. The basis vectors of l_p and l_r will be denoted by $\{\mathbf{e}^{[i]}\}$ and $\{\mathbf{f}^{[j]}\}$ respectively. A linear operator $A: l_p \to l_r$ is said to have the matrix representation (a_{ij}) if and only if

$$A(\xi) = \lim_n \sum_{1 \leq i \leq n} \left(\sum_{1 \leq j < \infty} a_{ij} \xi_j \right) \mathbf{f}^{[i]}$$

where the limit is taken with respect to the weak topology on l_r from l_s. The associated diagonal operator of A, denoted by D, is the operator with matrix representation $(\delta_{ij}a_{ij})$. Let $\boldsymbol{\alpha}=(\alpha_1, \ldots, \alpha_i, \ldots)$ be a sequence of scalars. $D_\alpha: l_p \to l_r$ will denote the linear operator which satisfies $D_\alpha(\xi)=(\alpha_1\xi_1, \ldots, \alpha_i\xi_i, \ldots)$. We call D_α a diagonal linear operator.

(1.1) LEMMA. *Let r be a real number such that $r > 1$. Then*

$$|x|^r + |y|^r \geqq 2|(x+y)/2|^r$$

for all complex numbers x and y; equality holds if and only if $x=y$.

Proof. This follows from the Hölder inequality:

$$|x+y| \leqq |x|+|y| \leqq (|x|^r+|y|^r)^{1/r}2^{r-1/r}.$$

One checks that equality holds throughout if and only if $x=y$.

(1.2) DEFINITION. A sign distribution on n places is a function $\sigma: \{1, 2, \ldots, n\}$ $\to \{1, -1\}$. Whenever it is desired to specify the number of places σ is defined on we write $\sigma^{[n]}$. Two sign distributions σ, τ are said to be equivalent if and only if $\sigma(k)=\tau(k)$ for all $k=1, 2, \ldots, n$ or $\tau(k)=-\tau(k)$ for all $k=1, 2, \ldots, n$. Sign distributions which are not equivalent are said to be distinct. Given n places, there are exactly 2^{n-1} equivalence classes of sign distributions.

If $\mathscr{S}$ and $\mathscr{T}$ are collections of 2^{n-1} distinct sign distributions then there is a mapping $\theta: \mathscr{S} \to \mathscr{T}$ so that σ and $\theta(\sigma)$ are equivalent for every $\sigma \in \mathscr{S}$.

(1.3) LEMMA. *Let $\mathscr{S}=\{\sigma_1, \ldots, \sigma_i, \ldots, \sigma_{2^{n-1}}\}$ be a collection of distinct sign distributions. Let $x_1, \ldots, x_n$ be complex numbers. Let $r > 1$. Then*

$$(1.3.1) \qquad \sum_{1 \leqq i \leqq 2^{n-1}} \left| \sum_{1 \leqq k \leqq n} \sigma_i(k)x_k \right|^r \geqq 2^{n-1} \max \{|x_1|^r, \ldots, |x_n|^r\}.$$

(1.3.2) *Equality holds if and only if at most one x_i is nonzero.*

Proof. It suffices to show

$$\sum_{1 \leqq i \leqq 2^{n-1}} \left| \sum_{1 \leqq k \leqq n} \sigma_i(k)x_k \right|^r \geqq 2^{n-1}|x_1|^r$$

where equality holds if and only if $x_2=x_3=\cdots=x_n=0$. If $\mathscr{T}=\{\tau_1, \ldots, \tau_{2^{n-1}}\}$ is any collection of 2^{n-1} distinct sign distributions on n places, then

$$
\begin{aligned}
\sum_{1 \leqq i \leqq 2^{n-1}} \left| \sum_{1 \leqq k \leqq n} \sigma_i(k)x_k \right|^r &= \sum_{1 \leqq i \leqq 2^{n-1}} \left| \sum_{1 \leqq k \leqq n} \theta(\sigma_i)(k)x_k \right|^r \\
&= \sum_{1 \leqq j \leqq 2^{n-1}} \left| \sum_{1 \leqq k \leqq n} \tau_j(k)x_k \right|^r
\end{aligned}
$$

(1.3.3)

where θ is the map referred to in (1.2). We now choose $\mathscr{T}$ to satisfy:

(a) τ_1 satisfies $\tau_1(1)=1$. Let $1 \leqq j < 2^{n-1}$. Assume that $\tau_1, \ldots, \tau_j$ have been chosen so that $\{\tau_1, \ldots, \tau_j\}$ is a collection of distinct sign distributions and so that $\tau_{j'}(1)=1$ for all $j'=1, 2, \ldots, j$.

(b) If j is an odd integer, define τ_{j+1} by setting $\tau_{j+1}(1)=1$ and $\tau_{j+1}(k)=-\tau_j(k)$ for $k=2,\ldots,n$.

(c) If j is an even integer, choose τ_{j+1} to be any sign distribution distinct from $\{\tau_1,\ldots,\tau_j\}$ which satisfies $\tau_{j+1}(1)=1$.

We now check that $\mathscr{T}$ is a collection of distinct sign distributions. If j is an even integer, the definition of τ_{j+1} guarantees that $\{\tau_1,\ldots,\tau_{j+1}\}$ is a collection of distinct sign distributions. If j is an odd integer, then τ_{j+1} is clearly distinct from τ_j. If for some $k<j$, τ_k and τ_{j+1} were equivalent, then we would have either that τ_{k+1} and τ_j are equivalent (when k is odd) or that τ_{k-1} and τ_j are equivalent (when k is even). In either case a contradiction results because $k+1\neq j$ and $k-1\neq j$.

If j is an odd integer, condition (b) in the definition of $\mathscr{T}$ gives

$$\left|\sum_{1\leq k\leq n}\tau_{j+1}(k)x_k\right|^r = \left|x_1-\sum_{2\leq k\leq n}\tau_j(k)x_k\right|^r.$$

From Lemma 1.1 we get

$$\left|\sum_{1\leq k\leq n}\tau_j(k)x_k\right|^r+\left|\sum_{1\leq k\leq n}\tau_{j+1}(k)x_k\right|^r$$
$$= \left|x_1+\sum_{2\leq k\leq n}\tau_j(k)x_k\right|^r+\left|x_1-\sum_{2\leq k\leq n}\tau_j(k)x_k\right|^r \geq 2|x_1|^r$$

where equality holds if and only if

$$\sum_{2\leq k\leq n}\tau_j(k)x_k = 0.$$

Thus

$$\sum_{1\leq j\leq 2^{n-1}}\left|\sum_{1\leq k\leq n}\tau_j(k)x_k\right|^r = \sum_{1\leq l\leq 2^{n-2}}\left(\left|\sum_{1\leq k\leq n}\tau_{2l-1}(k)x_k\right|^r+\left|\sum_{1\leq k\leq n}\tau_{2l}(k)x_k\right|^r\right)$$
$$\geq \sum_{1\leq l\leq 2^{n-2}}|x_1|^r = 2^{n-1}|x_1|^r,$$

where equality holds if and only if

(1.3.4)
$$\sum_{2\leq k\leq n}\tau_{2l-1}^{[n]}(k)x_k = 0$$

for all $l=1,2,\ldots,2^{n-2}$. By (1.3.3) we get

(1.3.5)
$$\sum_{1\leq i\leq 2^{n-1}}\left|\sum_{1\leq k\leq n}\sigma_i(k)x_k\right|^r \geq 2^{n-1}|x_1|^r$$

where equality holds if and only if (1.3.4) holds. It remains to show that (1.3.4) implies $x_2=x_3=\cdots=x_n=0$. Define $\{\nu_1^{[n-1]},\ldots,\nu_{2^{n-2}}^{[n-1]}\}$ to be a collection of 2^{n-2} sign distributions on $n-1$ places by setting $\nu_i^{[n-1]}(k)=\tau_{2l-1}(k+1)$ for $k=1,\ldots,$ $n-1$. Since $\tau_{2l-1}(1)=1$ for all $l=1,2,\ldots,2^{n-2}$ and since $\{\tau_{2l-1}\}_{1\leq l\leq 2^{n-2}}$ is a

collection of distinct sign distributions on n places, the collection of sign distributions on $n-1$ places $\{v_l^{[n-1]}\}_{1 \leq l \leq 2^{n-2}}$ must also be distinct. Reformulate (1.3.4) as

$$\sum_{2 \leq k \leq n} v_l^{[n-1]}(k-1)x_k = 0 \quad \text{for } l = 1, \ldots, 2^{n-2}.$$

Apply (1.3.5) to get

$$0 = \sum_{1 \leq l \leq 2^{n-2}} \left| \sum_{2 \leq k \leq n} v_l^{[n-1]}(k-1)x_k \right|^r \geq \max \{|x_2|^r, \ldots, |x_n|^r\}.$$

Thus $x_2 = x_3 = \cdots = x_n = 0$.   Q.E.D.

§ 2

The following section computes the operator sup norm of a diagonal linear operator $D_\alpha : l_p \to l_r$.

(2.1) DEFINITION. Let $1 \leq p, r \leq \infty$. Define

$$\begin{aligned} g(p, r) &= \infty && \text{if } 1 \leq p \leq r \leq \infty, \\ &= pr/(p-r) && \text{if } 1 \leq r < p < \infty, \\ &= 1 && \text{if } 1 \leq r < \infty \text{ and } p = \infty. \end{aligned}$$

Whenever convenient we shall write g in place of $g(p, r)$.

(2.2) PROPOSITION. *Let $D_\alpha : l_p \to l_r$ be a diagonal linear operator. Then $\|D_\alpha\| = l_g(\alpha)$.*

Proof. When $1 \leq p \leq r \leq \infty$, the proposition is clear. We indicate the proof when $1 \leq r < p < \infty$. Since $p/r > 1$, the Hölder inequality gives

$$\|D_\alpha(\xi)\|^r \leq \left(\sum_i |\alpha_i|^{pr/(p-r)} \right)^{(p-r)/p} \left(\sum_i |\xi_i|^p \right)^{r/p}.$$

Thus

$$\|D_\alpha(\xi)\| \leq l_g(\alpha) l_p(\xi)$$

and so,

$$\|D_\alpha\| \leq l_g(\alpha).$$

Define $\xi = (\xi_1, \ldots, \xi_i, \ldots)$ by setting $\xi_i = |\alpha_i|^{(r/(p-r))}$. Direct verification proves that

$$l_g(\alpha) l_p(\xi) = l_r(D_\alpha(\xi)) \leq \|D_\alpha\| l_p(\xi).$$

Thus $l_g(\alpha) \leq \|D_\alpha\|$. Therefore $\|D_\alpha\| = l_g(\alpha)$. The remaining case is proved similarly.

(2.3) DEFINITION. Let $A = (a_{ij})$ be a $Z \times Z$ matrix. We say that A is a diagonal block matrix determined by $\{m_k, n_k\}$ if and only if the sequence $\{m_k, n_k\}$ satisfies $1 = m_1 \leq n_1 < m_2 \leq n_2 \cdots < m_k \leq n_k \cdots$ where $a_{ij} \neq 0$ only when i and j satisfy: there is a k for which $m_k \leq i, j \leq n_k$. We say that A is supported on the (m_k, n_k) block if and only if $a_{ij} \neq 0$ only when i and j satisfy: there is a k for which $m_k \leq i, j \leq n_k$. The $Z \times Z$ submatrix of A which is supported on the (m_k, n_k) block is denoted by

$A^{[k]}$. Computations for diagonal block operators are used in this paper to prove (2) of Corollary (3.9).

(2.4) DEFINITION. The support of a vector ξ in l_p is the set of all indices i for which $\xi_i \neq 0$. A sequence of vectors $\xi^{[k]}$ in l_p is said to be disjointly supported if and only if the family of supports of $\xi^{[k]}$ is disjoint. If $\{\xi^{[k]}\}$ is a sequence of disjointly supported vectors then

$$l_p\left(\sum_k \xi^{[k]}\right) = \left(\sum_k |l_p(\xi^{[k]})|^p\right)^{1/p}.$$

(2.5) PROPOSITION. *Let $1 \leqq p, r \leqq \infty$. Let D be a diagonal block matrix representing a linear operator from l_p to l_r. Given a sequence $1 = m_1 \leqq n_1 < \cdots < m_k \leqq n_k \cdots$, define $D^{[k]}$ as in (2.3). Let δ_k denote the operator* sup *norm of $D^{[k]}$. Then*

$$\|D\| = l_g(\delta_1, \ldots, \delta_2, \ldots).$$

Proof. It suffices to indicate the proof when $1 \leqq r < p < \infty$. For each $\xi \in l_p$ where $\xi = (\xi_1, \ldots, \xi_i, \ldots)$ define $\xi^{[k]} = (0, \ldots, 0, \xi_{m_k}, \ldots, \xi_i, \ldots, \xi_{n_k}, 0, \ldots)$. Direct verification shows $D(\xi) = \sum_k D_k(\xi^{[k]})$. And since $\{D_k(\xi^{[k]})\}$ is a disjointly supported sequence,

$$\|D(\xi)\|^r = \sum_k \|D_k(\xi^{[k]})\|^r.$$

Clearly, $l_r(D_k(\xi^{[k]})) \leqq \delta_r l_p(\xi^{[k]})$. Since $(p)/r > 1$, the Hölder inequality gives

$$\|D(\xi)\|^r \leqq \sum_k \delta_k^r l_p(\xi^{[k]})^r \leqq \left(\sum_k |\delta_k^r|^{p/(p-r)}\right)^{(p-r)/p}\left(\sum_k |l_p(\xi^{[k]})^r|^{p/r}\right)^{r/p}.$$

Thus

$$\|D(\xi)\| \leqq l_g(\delta_1, \ldots, \delta_k, \ldots)l_p(\xi),$$

i.e. $\|D\| \leqq l_g(\delta)$. To prove the reverse direction of the inequality, let $\xi^{[k]} \in l_p$ satisfy $l_p(\xi^{[k]}) = 1$ where $\xi^{[k]}$ is supported on those indices i such that $m_k \leqq i \leqq n_k$ and where $\|D^{[k]}(\xi^{[k]})\| = \delta_k$. Define $\zeta = (\zeta_1, \ldots, \zeta_i, \ldots)$ by setting

$$\zeta_i = |\delta_k|^{r/(p-r)}\xi_1^{[k]} \quad \text{whenever } m_k \leqq i \leqq n_k,$$
$$= 0 \qquad\qquad\quad \text{otherwise.}$$

It is easy to check that $\|D(\zeta)\| = l_g(\delta)l_p(\zeta)$. Thus $\|D\| \geqq l_g(\delta)$.

§3

Let $A: l_p \to l_r$ be a linear operator with matrix representation (a_{ij}). By the diagonal linear operator associated with A we mean the linear operator (if it exists) which has the matrix representation $(\delta_{ij}a_{ij})$. We will show that if A is a bounded operator then the diagonal linear operator associated with A (denoted by D) exists and satisfies $\|D\| \leqq \|A\|$. Under certain conditions on p and r, if $\|D\| = \|A\|$ then $D = A$.

(3.1) LEMMA. *Let* $1 < r < p < \infty$. *Let* $A: l_p^n \to l_r^n$ *be a linear operator with matrix representation* (a_{ij}). *Let* $\mathscr{S} = \{\sigma_1, \ldots, \sigma_{2^{n-1}}\}$ *be a collection of* 2^{n-1} *distinct sign distributions on* n *places. For each* $j = 1, 2, \ldots, 2^{n-1}$ *define* $\xi^{[j]} \in l_p^n$ *by* $\xi_k^{[j]} = \sigma_j(k) a_{kk}^{r/(p-r)}$ *for* $k = 1, \ldots, n$. *Suppose* $A \neq D$. *Then there is a* j *for which* $\|A(\xi^{[j]})\| > l_p(\xi^{[j]}) \|D\|$.

Proof. Suppose not. Then $\|A(\xi^{[j]})\| \leq l_p(\xi^{[j]}) \|D\|$ for all $j = 1, 2, \ldots, n$. By (2.2),

$$\left(\sum_{1 \leq i \leq n} \left| \sum_{1 \leq k \leq n} a_{ik} \sigma_j(k) a_{kk}^{r/(p-r)} \right|^r \right)^{1/r} \leq \left(\sum_{1 \leq k \leq n} |\sigma_j(k) a_{kk}^{r/(p-r)}|^p \right)^{1/p} l_g(a_{11}, \ldots, a_{nn})$$

for all $j = 1, 2, \ldots, 2^{n-1}$. Since $|\sigma_j(k)| = 1$, raising both sides of the 2^{n-1} inequalities to the rth power and summing gives:

(3.1.1)
$$\sum_{1 \leq j \leq 2^{n-1}} \sum_{1 \leq i \leq n} \left| \sum_{1 \leq k \leq n} a_{ik} \sigma_j(k) a_{kk}^{r/(p-r)} \right|^r$$
$$\leq 2^{n-1} \left(\sum_{1 \leq k \leq n} |a_{kk}|^{pr/(p-r)} \right)^{r/p} \left(\sum_{1 \leq k \leq n} |a_{kk}|^{pr/(p-r)} \right)^{(p-r)/p}$$
$$= 2^{n-1} \sum_{1 \leq k \leq n} |a_{kk}|^{pr/(p-r)}.$$

Apply Lemma (1.3) (for each fixed i) to get

(3.1.2)
$$\sum_{1 \leq j \leq 2^{n-1}} \sum_{1 \leq i \leq n} \left| \sum_{1 \leq k \leq n} a_{ik} \sigma_j(k) a_{kk}^{r/(p-r)} \right|^r$$
$$= \sum_{1 \leq i \leq n} \sum_{1 \leq j \leq 2^{n-1}} \left| \sum_{1 \leq k \leq n} a_{ik} \sigma_j(k) a_{kk}^{r/(p-r)} \right|^r$$
$$\geq 2^{n-1} \sum_{1 \leq i \leq n} \max \{|a_{i1} a_{11}^{r/(p-r)}|^r, \ldots, |a_{ik} a_{kk}^{r/(p-r)}|^r \cdots\}$$
$$\geq 2^{n-1} \sum_{1 \leq i \leq n} |a_{ii} a_{ii}^{r/(p-r)}|^r = 2^{n-1} \sum_{1 \leq i \leq n} |a_{ii}|^{pr/(p-r)}$$

where equality holds if and only if $|a_{ik} a_{kk}^{r/(p-r)}|^r = 0$ whenever $i \neq k$. Combine (3.1.1) and (3.1.2) to get

$$\sum_{1 \leq j \leq 2^{n-1}} \sum_{1 \leq i \leq n} \left| \sum_{1 \leq k \leq n} a_{ik} \sigma_j(k) a_{kk}^{r/(p-r)} \right|^r = 2^{n-1} \sum_{1 \leq i \leq n} |a_{ii}|^{pr/(p-r)}$$

so that (3.1.2) gives that $|a_{ik} a_{kk}^{r/(p-r)}|^r = 0$ whenever $i \neq k$.

Assuming that $a_{kk} \neq 0$ for all k, this gives $a_{ik} = 0$ for $i \neq k$. Hence $A = D$ contrary to hypothesis. Hence there is a j for which

$$\|A(\xi^{[j]})\| \leq l_p(\xi^{[j]}) \|D\|.$$

Suppose $a_{kk} = 0$ for some k. By suitable rearrangement of the $\mathbf{e}^{[i]}$'s and $\mathbf{f}^{[j]}$'s we may assume the existence of an integer m for which $a_{kk} = 0$ if and only if $m < k \leq n$. Let $B: l_p^m \to l_r^m$ be the operator whose matrix representation (b_{ij}) satisfies $b_{ij} = a_{ij}$ for all $1 \leq i, j \leq m$. It is easy to check that

$$\|A(\xi^{[j]})\| \geq \|B(\xi^{[j]})\|$$

for all j. If E denotes the diagonal operator associated with B, the case which we have just proved above gives the existence of a j for which $\|B(\xi^{[j]})\| > \|E\|$. Since it is clear that $\|E\| = \|D\|$, we get $\|A(\xi^{[j]})\| > \|D\|$, as required.   Q.E.D.

(3.2) LEMMA. *Let $1 < r < \infty$. Let $A: l_\infty^n \to l_r^n$. Let $\mathscr{S} = \{\sigma_1, \ldots, \sigma_{2^{n-1}}\}$ be a collection of 2^{n-1} distinct sign distributions on n places. For $j = 1, \ldots, 2^{n-1}$, define $\xi^{[j]} \in l_\infty^n$ by setting $\xi_k^{[j]} = \sigma_j(k)$ for $k = 1, \ldots, n$. Suppose $A \neq D$. Then there is a j for which $\|A(\xi^{[j]})\| > l_\infty(\xi^{[j]})\|D\|$.*

Proof. The proof proceeds analogously as in Lemma 1.

(3.3) THEOREM. *Let $1 \leqq p, r \leqq \infty$. Let $A: l_p^n \to l_r^n$ be a linear operator. Let D be the associated diagonal linear operator. Then*

(1) $\|A\| \geqq \|D\|$;

(2) *when p, r satisfy $1 < r < p \leqq \infty$ or $r = 1$ and $1 < p < \infty$, then equality in (1) holds if and only if $A = D$.*

Proof. The case $1 < r < p \leqq \infty$ was covered by (3.1) and (3.2). When $r = 1$ and $1 < p < \infty$, we observe that $\|A\| = \|A'\|$ where $A': l_\infty^n \to l_q^n$ is the adjoint map of A. If D' denotes the associated diagonal of A' then D' is the adjoint of D and so $\|D\| = \|D'\|$. Since $1 < q < \infty$, Lemma (3.2) gives $\|A'\| \geqq \|D'\|$ where equality holds if and only if $A' = D'$. Thus $\|A\| \geqq \|D\|$ where equality holds if and only if $A = D$.

It remains to prove: $\|A\| \geqq \|D\|$ for the case $1 \leqq p \leqq r \leqq \infty$ and the case $p = \infty$, $r = 1$. The latter case is proved by defining $\xi^{[j]}$ as in Lemma (3.2) and proceeding as in Lemma (3.2) where the triangle inequality is used in place of Lemma (1.1).

When $1 \leqq p \leqq r \leqq \infty$, then $g(p, r) = \infty$ and so $\|D\| = l_\infty(a_{11}, \ldots, a_{kk}, \ldots, a_{nn})$. Let m be the index for which $|a_{mm}| = l_\infty(a_{11}, \ldots, a_{kk}, \ldots, a_{nn})$. Let $\xi \in l_p$ be defined by $\xi_i = \delta_{im}$. It is easy to check that

$$\|A(\xi)\| \geqq |a_{mm}| = \|D\| l_p(\xi).$$

Hence $\|A\| \geqq \|D\|$.   Q.E.D.

(3.4) REMARKS. We give examples to show that Theorem (3.3) cannot be improved. If we take l_∞^2 and l_1^2 to be vector spaces over the reals, the linear operator $A: l_\infty^2 \to l_1^2$ with matrix representation

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

is easily seen to have norm 2 which is also the norm of the associated diagonal of D. But $A \neq D$.

When $1 \leqq p \leqq r \leqq \infty$, the linear operator $A: l_p^3 \to l_r^3$ with the matrix representation

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

has norm 1. The associated diagonal D also has norm 1 but $A \neq D$.

(3.5) DEFINITION. Let $A = (a_{ij})$ be a $t \times t$ matrix. Let $\{m_k, n_k\}$ be a sequence of indices satisfying

$$1 = m_1 \leqq n_1 \leqq m_2 \leqq n_2 < \cdots < m_u \leqq n_u = t.$$

By the associated diagonal block operator of A determined by $\{m_k, n_k\}$ (which we denote as D) we mean the diagonal block operator with matrix representation (d_{ij}) where

$$d_{ij} = a_{ij} \quad \text{whenever } m_k \leqq i, j \leqq n_k, \text{ for some } k \text{ where } 1 \leqq k \leqq u,$$
$$= 0 \quad \text{otherwise.}$$

(3.6) LEMMA. *Let A, D, and $\{m_k, n_k\}$ be as described in* (3.5). *Then* $\|A\| \geqq \|D\|$.

Proof. Let $D^{[k]}$ be as defined in (2.3). Let $\delta_k = \|D^{[k]}\|$. By (2.5), $\|D\| = l_g(\delta_1, \ldots, \delta_k, \ldots, \delta_u)$. We will show that a linear operator $B: l_p^u \to l_r^u$ exists so that

(3.6.1) (a) $\|A\| \geqq \|B\|$ and (b) the associated diagonal of B which we denote as E has δ_k on its kth diagonal entry.

If (3.6.1) were proved, then $\|A\| \geqq \|E\|$. But

$$\|E\| = l_g(\delta_1, \ldots, \delta_u) = \|D\|.$$

Thus the lemma immediately follows.

By a suitable rearrangement of the $\mathbf{e}^{[i]}$'s and $\mathbf{f}^{[j]}$'s we may assume that the matrix representation of D is supported on (m'_k, n'_k)-blocks for some sequence

$$1 = m'_1 \leqq n'_1 < m'_2 \leqq n'_2 < \cdots < m'_z \leqq n'_z \leqq t$$

where $\{m'_k, n'_k\}$ satisfies $m'_{k+1} = n'_k + 1$ for all $1 \leqq k \leqq z$. For the present we assume that $n'_z = t$.

Let v, w be integers satisfying $1 \leqq v, w \leqq z$. Define $A^{[v,w]}$ to be the linear operator with matrix representation $(a_{ij}^{[v,w]})$ where

$$a_{ij}^{[v,w]} = a_{ij} \quad \text{if } m_v \leqq i \leqq n_v \text{ and } m_w \leqq j \leqq n_w,$$
$$= 0 \quad \text{otherwise.}$$

Clearly, $A^{[k,k]} = D^{[k]}$. Choose $\mu^{[k]} \in l_p^z$ to satisfy:
  (a) $\mu^{[k]}$ is supported on the indices i such that $m_k \leqq i \leqq n_k$.
  (b) $\|\mu^{[k]}\| = 1$.
  (c) $\|D^{[k]}(\mu^{[k]})\| = \|D^{[k]}\| = \delta_k$.
Choose $\mathbf{v}^{[k]} \in l_s^z$ to satisfy
  (a) $\mathbf{v}^{[k]}$ is supported on the indices j such that $m_k \leqq j \leqq n_k$.
  (b) $\|\mathbf{v}^{[k]}\| = 1$.
  (c) $(D^{[k]}(\mu^{[k]}), \mathbf{v}^{[k]}) = \delta_k$.
Define $B: l_p^z \to l_r^z$ by setting

$$(B(\xi), \zeta) = \sum_{1 \leqq v \leqq z} \zeta_v \left(\sum_{1 \leqq k \leqq z} A^{[v,k]}(\xi_k \mu^{[k]}), \mathbf{v}^{[v]} \right)$$

for all $\zeta \in l_s^z$. If b_{ii} is the ith diagonal entry of S, we have by definition,

$$b_{ii} = (B(e^{[i]}), f^{[i]}) = (A^{[i,i]}(\mu^{[i]}), \nu^{[i]})$$
$$= (D^{[i]}(\mu^{[i]}), \nu^{[i]}) = \delta_i.$$

This proves (3.6.1)(b). To prove (3.6.1)(a), we compute that when $\xi \in l_p^z$ and $\zeta \in l_s^z$ satisfy $\|\xi\| = \|\zeta\| = 1$, then

$$\left\| \sum_{1 \leq k \leq z} \xi_k \mu^{[k]} \right\| = \left\| \sum_{1 \leq k \leq z} \zeta_k \nu^{[k]} \right\| = 1$$

(because the μ's and ν's are disjointly supported) so that

$$\|A\| \geq \left| \left(A\left(\sum_{1 \leq k \leq z} \xi_k \mu^{[k]} \right), \sum_{1 \leq w \leq z} \zeta_w \nu^{[w]} \right) \right|$$
$$= \left| \left(\sum_{1 \leq v \leq z} \sum_{1 \leq k \leq z} A^{[v,k]}(\xi_k \mu^{[k]}), \sum_{1 \leq w \leq z} \zeta_w \nu^{[w]} \right) \right|$$
$$= \left| \sum_{1 \leq v \leq z} \left(\sum_{1 \leq k \leq z} A^{[v,k]}(\xi_k \mu^{[k]}), \zeta_v \nu^{[u]} \right) \right|$$
$$= |(B(\xi), \zeta)|.$$

Here we use the easily verified fact that

$$\left(\sum_{1 \leq k \leq z} A^{[v,k]}(\xi_k \mu^{[k]}), \zeta_w \nu^{[w]} \right) = 0$$

when $w \neq v$. This shows (3.6.1)(a).

To remove the hypothesis $n_z' = t$, assume $n_z' < t$ and define $m_{z+1}' = n_{z+1}'$, $n_{z+1}' = t$. Let $B_1 : l_p^{z+1} \to l_r^{z+1}$ be defined in a manner similar to B where z is replaced by $z+1$. If E_1 is the associated diagonal of B_1 then the case we have just proved and (3.3) show that $\|A\| \geq \|B_1\| \geq \|E_1\| \geq \|D\|$.   Q.E.D.

If p and r satisfy the conditions in (2) of (3.3), it is not known whether or not $\|A\| = \|D\|$ implies $A = D$.

(3.7) THEOREM. *Let $1 \leq p, r \leq \infty$. Let $A : l_p \to l_r$. Let $\{m_k, n_k\}$ be a sequence so that*

$$1 = m_1 \leq n_1 < m_2 \leq n_2 < \cdots < m_k \leq n_k \cdots.$$

Let D be the diagonal block operator of A determined by $\{m_k, n_k\}$. Then $\|A\| \geq \|D\|$.

Proof. Write $A_u = \pi_{n_u} A \pi_{n_u}$. By (2.5),

$$\|D\| = l_g(d_1, \ldots, d_k, \ldots) = \lim_u l_g(d_1, \ldots, d_u) = \lim_u \|D_u\|$$

where D_u is the diagonal block operator of A_u determined by $\{m_k, n_k\}$. By (3.6), $\|A_u\| \geq \|D_u\|$. Since

$$\|A\| = \lim_u \|A_u\|,$$

we get $\|A\| \geq \|D\|$.   Q.E.D.

(3.8) PROPOSITION. *Let* $1 \leq p, r \leq \infty$. *Let* $\{m_k, n_k\}$ *be a sequence of indices satisfying*

$$1 = m_1 \leq n_1 < m_2 \leq n_2 < \cdots < m_k \leq n_k \cdots .$$

Let $A: l_p \to l_r$ *be a linear operator with matrix representation* (a_{ij}). *Let* D *be the diagonal block operator of* A *determined by* $\{m_k, n_k\}$. *Let* t *be an integer such that if* $B = \pi_{n_t} A \pi_{n_t}$ *and* E *is the associated diagonal block of* B, *then* $\|B\| > \|E\|$. *Then* $\|A\| > \|D\|$.

Proof. Let $d_k = \|D^{[k]}\|$. By (2.5),

$$\|D\| = l_g(d_1, \ldots, d_k, \ldots)$$

$$= \left(\|E\|^g + \sum_{k>t} d_k^g \right)^{1/g}$$

$$< \left(\|B\|^g + \sum_{k>t} d_k^g \right)^{1/g}.$$

Let $m_1' = 1$, $n_1' = n_t$, $m_i' = m_{t+i-1}$, $n_i' = n_{t+i-1}$. If D_0 is the diagonal block operator of A determined by $\{m_k', n_k'\}$, then (3.7) gives $\|A\| \geq \|D_0\|$. By (2.5) and the above,

$$\|D_0\| = \left(\|B\|^g + \sum_{k>t} d_k^g \right)^{1/g} > \|D\|.$$

Hence $\|A\| > \|D\|$.

(3.9) COROLLARY. *Let* $1 \leq p, r \leq \infty$. *Let* $A: l_p \to l_r$ *be a linear operator and let* D *be the associated diagonal operator of* A. *Then*

(1) $\|A\| \geq \|D\|$.

(2) *If* p, r *satisfies* $1 < r < p \leq \infty$ *or* $r = 1$ *and* $1 \leq p < \infty$, *then equality in* (1) *holds if and only if* $A = D$.

Proof. (1) follows from (3.7) above. To show (2), suppose $A \neq D$. Choose an integer K sufficiently large so that if $B = \pi_K A \pi_K$ then $B \neq E$ where E is the associated diagonal of B. By (3.3), $\|B\| > \|E\|$. By (3.8), this implies $\|A\| > \|D\|$.   Q.E.D.

An analogous assertion to (2) of the corollary above for all diagonal block operators, on a given operator A, determined by $\{m_k, n_k\}$ can be proved as above provided Lemma (3.6) can be improved so as to include an analogous assertion to (2) of (3.3).

§ 4

This section computes the nuclear norm of diagonal linear operators $E: l_q \to l_s$. We follow the definitions of nuclear operator and nuclear norm given by A. Pietsch [2]. If $\mathcal{B}_1, \mathcal{B}_2$ are Banach spaces, $\mathcal{N}(\mathcal{B}_1, \mathcal{B}_2)$ will denote the Banach space of all nuclear operators from $\mathcal{B}_1$ to $\mathcal{B}_2$ and ν will denote the nuclear norm. Let $\mathcal{B}_1'$ denote the Banach dual of $\mathcal{B}_1$. $\mathcal{B}_1' \otimes \mathcal{B}_2$ will denote the algebraic tensor product, γ will denote the greatest crossnorm as given in Grothendieck [1], and $\mathcal{B}_1' \widehat{\otimes}_\gamma \mathcal{B}_2$ will denote the completion of $\mathcal{B}_1' \otimes \mathcal{B}_2$ under the norm γ. From A. Grothendieck

[1, p. 165] it is known that if either $\mathcal{B}_1'$ or $\mathcal{B}_2$ has the approximation property, then there is a one-one linear map of $\mathcal{B}_1' \otimes_\gamma \mathcal{B}_2$ into the space of bounded operators from $\mathcal{B}_1$ to $\mathcal{B}_2$ which is denoted as $\mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$. It is easy to check that in this case the one-one linear map induces an isometric isomorphism of $\mathcal{B}_1' \otimes_\gamma \mathcal{B}_2$ onto $\mathcal{N}(\mathcal{B}_1, \mathcal{B}_2)$. Since, in this section, we shall restrict $\mathcal{B}_1 = l_q$, $\mathcal{B}_2 = l_s$ where $1 \leq q$, $s \leq \infty$, we may assume the existence of the isometric isomorphism and use $\mathcal{B}_1' \otimes_\gamma \mathcal{B}_2$ and $\mathcal{N}(\mathcal{B}_1, \mathcal{B}_2)$ interchangeably.

The following is well known (Proposition 2, §1, no. 1 of [1]):

(4.1) THEOREM. *The map $L: (\mathcal{B}_1' \otimes_\gamma \mathcal{B}_2)' \to \mathcal{L}(\mathcal{B}_1', \mathcal{B}_2')$ defined, for each $Q \in (\mathcal{B}_1' \otimes_\gamma \mathcal{B}_2)'$, by setting L_Q to be the operator satisfying $(L_Q(x'), y) = Q(x' \otimes y)$ for all $x' \otimes y \in \mathcal{B}_1' \otimes_\gamma \mathcal{B}_2$ is an isometric isomorphism.*

(4.2) PROPOSITION. *Let $1 \leq q$, $s \leq \infty$. Let $B: l_q^n \to l_s^n$ be a nuclear operator with matrix representation (b_{ij}). Let E denote the associated diagonal of B with diagonal entries $(b_{11}, \ldots, b_{ii}, \ldots, b_{nn})$. Then*

$$\nu(E) = \sup \{|(E, Q)| : \text{where } \|Q\| \leq 1 \text{ and } L_Q \text{ is a diagonal operator}\}.$$

Proof. By (4.1),

$$\nu(E) = \gamma(E) = \sup \{|(E, Q)| : \text{where } \|Q\| \leq 1 \text{ and } Q \in (l_p^n \otimes_\gamma l_s^n)'\}$$
$$\geq \sup \{|(E, Q)| : \text{where } \|Q\| \leq 1 \text{ and } L_Q \text{ is a diagonal operator}\}.$$

To prove the reverse inequality, let L_Q be any element of $\mathcal{L}(l_p^n, l_r^n)$. Let (a_{ij}) be the matrix representation of L_Q and D be the associated diagonal of L_Q. By verification,

$$(E, Q) = \sum_{1 \leq i \leq n} a_{ii} b_{ii} = (E, D).$$

By (3.9), $\|D\| \leq \|L_Q\| = \|Q\|$ and so,

$$\gamma(E) = \sup \{|(E, Q)| : \|Q\| \leq 1 \text{ and } Q \in (l_p^n \otimes_\gamma l_s^n)'\}$$
$$\leq \sup \{|(E, D)| : \|D\| \leq 1 \text{ and } D \in \mathcal{L}(l_p^n, l_r^n) \text{ is a diagonal operator}\}.$$

(4.3) THEOREM. *Let $1 \leq q$, $s \leq \infty$. Define $h(q, s)$ to be the dual number of $g(p, r)$. Let $E: l_q^n \to l_s^n$ be a diagonal linear operator with diagonal entries $(b_{11}, \ldots, b_{nn})$. Then*

$$\nu(E) = l_{h(q,s)}(b_{11}, \ldots, b_{nn}).$$

Proof. Let $D: l_p^n \to l_r^n$ be a diagonal operator with diagonal entries $(a_{11}, \ldots, a_{nn})$. By (4.2) above,

$$\nu(E) = \sup \left\{ (E, D) = \sum_{1 \leq i \leq n} a_{ii} b_{ii} : \|D\| \leq 1 \text{ where } D \text{ is a diagonal operator} \right\}.$$

Hence ν is just the dual norm on E when E is regarded as being in the dual of the space of all bounded diagonal operators $D: l_p^n \to l_r^n$. Since (2.2) shows that the

latter space is isometrically isomorphic to l_g^n, the dual norm ν must be the l_h norm where h is the dual number of g.   Q.E.D.

Whenever convenient we will use h instead of $h(q, s)$. Note that

$$h(q, s) = 1 \qquad\qquad \text{if } 1 \leqq s \leqq q \leqq \infty,$$
$$= qs/(qs+q-s) \quad \text{if } 1 \leqq q < s < \infty,$$
$$= q/(q-1) \qquad\quad \text{if } s = \infty \text{ and } 1 < q < \infty,$$
$$= \infty \qquad\qquad\quad \text{if } s = \infty \text{ and } q = 1.$$

We will take the convention of setting $l_{h(q,s)} = c_0$ when $h(q, s) = \infty$.

(4.4) THEOREM. *Let* $1 \leqq q$, $s \leqq \infty$. *Let* $E: l_q \to l_s$ *be a diagonal linear operator with diagonal entries* $(b_{11}, \ldots, b_{ii}, \ldots)$. *Then* E *is a nuclear operator if and only if* $(b_{11}, \ldots, b_{ii}, \ldots) \in l_h$; *in fact,* $\nu(E) = l_h(b_{11}, \ldots, b_{ii}, \ldots)$.

Proof. Define $\pi_n: l_r \to l_r^n$ by setting $\pi_n(\boldsymbol{\xi}) = (\xi_1, \ldots, \xi_n)$ for each $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_i, \ldots) \in l_r$. Define $\iota_n: l_p^n \to l_p$ by setting $\iota_n(\boldsymbol{\xi}) = (\xi_1, \ldots, \xi_n, 0, \ldots)$ for each $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n) \in l_p^n$. Clearly $\|\pi_n\| = \|\iota_n\| = 1$. If E is nuclear then $\pi_n E \iota_n$ is nuclear and

$$\nu(\pi_n E \iota_n) \leqq \|\pi_n\| \nu(E) \|\iota_n\| = \nu(E).$$

By (4.3), $\nu(\pi_n E \iota_n) = l_h(b_{11}, \ldots, b_{nn})$. Hence,

$$l_h(b_{11}, \ldots, b_{ii}, \ldots) = \lim_n l_h(b_{11}, \ldots, b_{nn}) \leqq \nu(E).$$

Thus, if E is nuclear then $(b_{11}, \ldots, b_{ii}, \ldots) \in l_h$ (provided $h \neq \infty$) and

$$l_h(b_{11}, \ldots, b_{ii}, \ldots) \leqq \nu(E).$$

If $h = \infty$, a little more argument is needed to show that $(b_{11}, \ldots, b_{ii}, \ldots) \in c_0$. Suppose $(b_{11}, \ldots, b_{ii}, \ldots) \notin c_0$. Then E is not a compact operator. But every nuclear operator is compact; thus, $(b_{11}, \ldots, b_{ii}, \ldots) \in c_0$. Conversely, suppose $l_h(b_{11}, \ldots, b_{ii}, \ldots) < \infty$. Then the sequence $\{(b_{11}, \ldots, b_{nn}, 0, \ldots)\}_n$ is Cauchy in l_h. By (4.3) above, this means that the sequence of diagonal operators $E^{[n]}$ with diagonal entries $(b_{11}, \ldots, b_{nn}, 0, \ldots)$ is Cauchy in $\mathcal{N}(l_q, l_s)$. Since $\mathcal{N}(l_q, l_s)$ is Banach, $\{E^{[n]}\}$ converges to some nuclear operator which we claim can only be E. Note that the topology given by the nuclear norm is stronger than the topology of simple convergence. Indeed, it is easy to see that

$$\lim_n E^{[n]}(\xi_1, \ldots, \xi_i, \ldots) = (b_{11}\xi_1, \ldots, b_{ii}\xi_i, \ldots)$$

so that the operator which $\{E^{[n]}\}$ converges to in the topology of simple convergence is E. Therefore E is well defined, the sequence $\{E^{[n]}\}$ must also converge to E in the nuclear norm and so

$$\nu(E) = \lim_n \nu(E^{[n]}) = \lim_n l_h(b_{11}, \ldots, b_{nn}, 0, \ldots)$$
$$= l_h(b_{11}, \ldots, b_{ii}, \ldots).$$

(4.5) THEOREM. *Let $1 \leqq q$, $s \leqq \infty$. Let $B: l_q \to l_s$ be a nuclear operator with matrix representation (b_{ij}). Let E be the associated diagonal linear operator of B. Then*

(1) *E is a nuclear operator and*

(2) *$\nu(E) \leqq \nu(B)$.*

Proof. Let

$$Q_n: \mathcal{N}(l_q^n, l_s^n) \to \mathcal{N}(l_q^n, l_s^n)$$

be the operator which takes each nuclear operator to its associated diagonal: Let

$$P_n: \mathcal{L}(l_p^n, l_r^n) \to \mathcal{L}(l_p^n, l_r^n)$$

be the adjoint of Q_n. P_n takes each operator to its associated diagonal. By (3.9), $\|P_n\| = 1$. Since $\|Q_n\| = \|P_n\|$, we get $\|Q_n\| = 1$. For each $B \in \mathcal{N}(l_q, l_s)$,

$$\pi_n B \iota_n \in \mathcal{N}(l_q^n, l_s^n).$$

Let E_n be the associated diagonal of $\pi_n B \iota_n$. Then

$$
\begin{aligned}
\nu(E) &= l_h(b_{11}, \ldots, b_{ii}, \ldots) = \lim_n l_h(b_{11}, \ldots, b_{nn}, 0, \ldots) \\
&= \lim_n \nu(E_n) \leqq \lim_n \|Q_n\| \nu(\pi_n B \iota_n) \\
&= \lim_n \nu(\pi_n B \iota_n) \leqq \lim_n \|\pi_n\| \nu(B) \|\iota_n\| = \nu(B). \qquad \text{Q.E.D.}
\end{aligned}
$$

(4.6) THEOREM. *Let $1 \leqq q$, $s \leqq \infty$. Let $\{m_k, n_k\}$ be a sequence of indices satisfying $1 = m_1 \leqq n_1 \leqq m_2 \leqq n_2 < \cdots < m_k \leqq n_k \cdots$. Let $B: l_q \to l_s$ be a nuclear operator with matrix representation (b_{ij}). Let E be the associated diagonal block operator of B. Let $E^{[k]}$ be the (m_k, n_k) block of B as defined in (2.3). Denote $\nu(E^{[k]}) = \varepsilon_k$. Then*

(1) *E is a nuclear operator,*

(2) *$\nu(E) = l_h(\varepsilon_1, \ldots, \varepsilon_k, \ldots)$ and*

(3) *$\nu(E) \leqq \nu(B)$.*

The proof proceeds as in Theorems (4.4) and (4.5) where (2.5) and (3.8) are used instead of (2.2) and (3.9).

This result is included for future reference. It turns out that if B_1, B_2 are Banach spaces satisfying certain basis conditions a criterion can be given in terms of diagonal block operators in $B_1 \otimes_\gamma^\frown B_2$ to decide: When is $(B_1 \otimes_\gamma^\frown B_2)'$ isometrically isometric to $B_1' \otimes_\gamma^\frown B_2'$? Theorem (4.6) will then be used to illustrate the criterion.

BIBLIOGRAPHY

1. A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Mem. Amer. Math. Soc. No. 16 (1955).

2. A. Pietsch, *Nukleare lokalkonvexe Räume*, Akademie-Verlag, Berlin, 1965.

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